Data Assimilation

miércoles, 26 de agosto de 2020 01:35 p. m.

DESIGN MATRIX

$$P(\beta|X,y) \propto P(\beta)\cdot \mathcal{L}(\beta|X,y) \longrightarrow \mathcal{L}(\beta|X,y) \propto e^{-\frac{1}{2}\|y-X\beta\|^2_{Z^{-1}}}$$

OBSERVATIONS

$$P(\beta) \propto e^{-\frac{1}{2}\|\beta-\hat{\beta}\|^2_{Q^{-1}}}$$

$x \quad x \quad x^b \quad y \quad H$

$$P(\beta|X,y) \propto e^{-J(\beta)} \qquad J(\beta) = \frac{1}{2}\|\beta-\hat{\beta}\|^2_{Q^{-1}} + \frac{1}{2}\|y-X\beta\|^2_{Z^{-1}}$$

$B^{-1} \quad x \quad R^{-1}$

$$\nabla_\beta(J(\beta)) = Q^{-1}[\beta-\hat{\beta}] - X^TZ^{-1}[y-X\beta]$$

$$= [Q^{-1}+X^TZ^{-1}X]\beta - [Q^{-1}\hat{\beta}+X^TZ^{-1}y]$$

$$\beta^* = \arg\max_\beta P(\beta|X,y) \equiv \beta^* = \arg\min_\beta J(\beta)$$

$Z = \sigma^2 I$

$x^a \quad (B^{-1} + H^TR^{-1}H)^{-1} \quad B^{-1}x^b \quad H^TR^{-1}y$

$Z^{-1} = \frac{1}{\sigma^2} I$

$$\Rightarrow \beta^* = [Q^{-1}+X^TZ^{-1}X]^{-1}[Q^{-1}\hat{\beta}+X^TZ^{-1}y]$$

$$[Q^{-1}+X^TZ^{-1}X]\beta^* = Q^{-1}\hat{\beta}+X^TZ^{-1}y$$

(*) $P(\beta|X,y) \propto P(\beta)\cdot\mathcal{L}(\beta|X,y)$

$$e^{-\frac{1}{2}\hat{J}(\beta)} \qquad \hat{J}(\beta) = \|\beta-\hat{\beta}\|^2_{Q^{-1}} + \|y-X\beta\|^2_{Z^{-1}}$$ (*)

(*) EXPONENT

$$\beta^TQ^{-1}\beta - \underbrace{\hat{\beta}^TQ^{-1}\beta - \beta^TQ^{-1}\hat{\beta}}_{=} + \hat{\beta}^TQ^{-1}\hat{\beta} + y^TZ^{-1}y - [X\beta]^TZ^{-1}y - y^TZ^{-1}[X\beta] + [X\beta]^TZ^{-1}[X\beta]$$

$\propto$

$$\Rightarrow \beta^TQ^{-1}\beta - 2\beta^TQ^{-1}\hat{\beta} - 2\beta^TX^TZ^{-1}y + \beta^TX^TZ^{-1}X\beta$$

$$\Rightarrow \beta^T Q^{-1} \overset{\circ}{\beta} - 2\beta^T \underbrace{\left[Q^{-1} \hat{\beta} + X^T Z^{-1} y \right]}_{\left[Q^{-1} + X^T Z^{-1} X\right] \beta^*} + \overset{\circ}{\beta^T} X^T Z^{-1} X \beta$$

$$\Rightarrow \beta^T \left[Q^{-1} + X^T Z^{-1} X \right] \beta - 2\beta^T \left[Q^{-1} + X^T Z^{-1} X \right] \beta^* + \beta^{*T} \left[Q^{-1} + X^T Z^{-1} X \right] \beta^*$$

α

$$\alpha \Rightarrow \left[\beta - \beta^*\right]^T \left[Q^{-1} + X^T Z^{-1} X \right] \left[\beta - \beta^*\right]$$

$$P(\beta \mid X, y) \propto e^{-\frac{1}{2}\left[\beta-\beta^*\right]^T \left[Q^{-1} + X^T Z^{-1} X\right]\left[\beta - \beta^*\right]}$$

$$= e^{-\frac{1}{2} \|\beta - \beta^*\|^2_{\left[Q^{-1} + X^T Z^{-1} X\right]^{-1}}}$$

(1) $\beta^* = \left[Q^{-1} + X^T Z^{-1} X\right]^{-1} \left[Q^{-1}\hat{\beta} + X^T Z^{-1} y\right]$

(2) $A_{\beta^*} = \left[Q^{-1} + X^T Z^{-1} X\right]^{-1}$

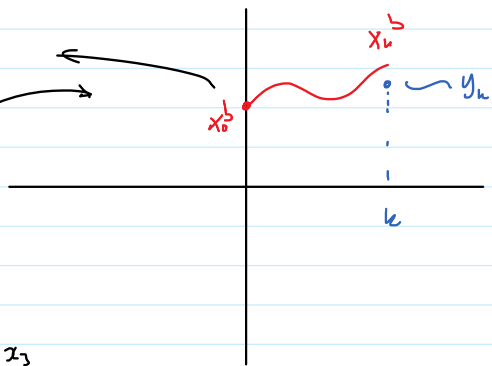
$\left(Q^{-1} + \frac{1}{\sigma^2} X^T X\right)^{-1} \Leftarrow \frac{1}{\sigma^2} I$

(+) ENSEMBLE KALMAN FILTER

(+) SEQUENTIAL DATA ASSIMILATION

$$\frac{dx}{dt} = f(t, x)$$

x_0



(+) FORECAST STATS — MODEL ERROR

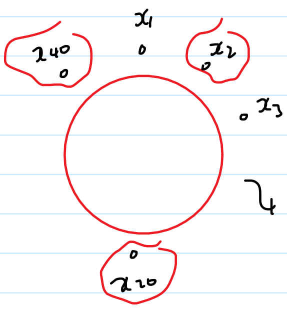
$$x_k^b = \mathcal{M}_{t_{k-1} \to t_k}(x_{k-1}^b) + q_k$$

$$\Rightarrow y_k = H_k(x_k) + \varepsilon_k$$

OBSERVATIONAL OPERATOR

x_k

$\varepsilon_k \sim \mathcal{N}(0_m, R_k)$



$\rightarrow H_k(x_k) = y_k$

$H(x) \; \mathbb{R}^n \Rightarrow \mathbb{R}^m$

$$x_k = \begin{bmatrix} x_{k,1} \\ x_{k,20} \\ x_{k,40} \end{bmatrix}$$

$$y_k = H_k(x_k^*) + \varepsilon_k$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ 0x_1 + 0x_2 + 1x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

PRIOR

?

$x_k^b \quad B_k$

$x_k \in \mathbb{R}^{n \times 1}$ — # TOTAL PUNTOS × # TOTAL NIVELES × # TOTAL VARIABLES

$y_k \quad R_k \quad$ OBS

$H_k \in \mathbb{R}^{m \times n}$

$$\underbrace{\varepsilon_k^b = x_k - x_k^b \sim \mathcal{N}(0, B_k)}_{\text{PRIORI}}$$

$$H_k \in \mathbb{R}^{m \times n}$$

$$\underbrace{\varepsilon_k^o = y_k - H_k x_k \sim \mathcal{N}(0, R_k)}_{\text{LIKELIHOOD}}$$

$$P(x_k \mid y_k) \propto P(x_k) \cdot \mathcal{L}(x_k \mid y_k)$$

OKIO $\underline{\underline{k}}$

$$\propto e^{-\frac{1}{2}\|x - x^b\|^2_{B^{-1}} - \frac{1}{2}\|y - Hx\|^2_{R^{-1}}}$$

$$\propto e^{-J(x)}$$

$$\underbrace{J(x) = \frac{1}{2}\|x - x^b\|^2_{B^{-1}} + \frac{1}{2}\|y - Hx\|^2_{R^{-1}}}$$

THREE DIMENSIONAL VARIATIONAL COST FUNCTION

3D-VAR

RIKKEU

$$E(\varepsilon_k^b \cdot \varepsilon_k^{o\,T}) = 0 \in \mathbb{R}^{n \times m}$$

NOT CORRELATED ERRORS FROM DIFFERENT SOURCES

ANALYSIS STATE

$$x^a = \arg\max_x P(x \mid y) \equiv x^a = \arg\min_x J(x)$$

POSTERIOR MODE

$$\Rightarrow \nabla_x(J(x)) = [B^{-1} + H^T R^{-1} H]\, x - [B^{-1} x^b + H^T R^{-1} y]$$

$$\nabla_x(J(x^a)) = 0 \Rightarrow x^a = [B^{-1} + H^T R^{-1} H]^{-1} [B^{-1} x^b + H^T R^{-1} y]$$

ANALYSIS STATE

$$\nabla^2_{xx}(J(x)) = B^{-1} + H^T R^{-1} H$$

$$\Rightarrow A = [\nabla^2_{xx}(J(x^a))]^{-1} = [B^{-1} + H^T R^{-1} H]^{-1}$$

$$\underbrace{x - x^b \sim \mathcal{N}(0, B)} \quad + \quad \underbrace{y - Hx \sim \mathcal{N}(0, R)} \quad \longrightarrow \quad \underbrace{x - x^a \sim \mathcal{N}(0, A)}$$

$x - x^b \sim N(0, B)$ $\qquad$ $y - Hx \sim N(0, R)$ $\qquad$ $x - x^a \sim N(0, A)$

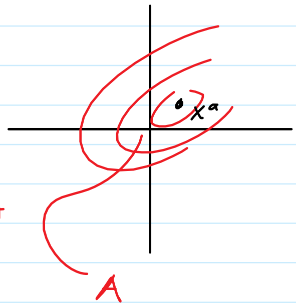
PRIOR DISTRIBUTION $\qquad$ DATA ERROR DISTRIBUTION $\qquad$ POSTERIOR DISTRIBUTION

HYPER PARAMETERS

$$x^a = [B^{-1} + H^T R^{-1} H]^{-1} [B^{-1} x^b + H^T R^{-1} y]$$

$$A = [B^{-1} + H^T R^{-1} H]^{-1}$$

POSTERIOR UNCERTAINT



(F) $\quad x \sim N(x^b, B) \Rightarrow x^{b(e)} \sim N(x^b, B)$ } NORMAL ASSUMPTIONS

e-th model realization

$$\mathbb{E}(\bar{x}^b) = \mathbb{E}\left(\frac{1}{N}\sum_{e=1}^{N} x^{b(e)}\right) = \frac{1}{N}\mathbb{E}\left(\sum_{e=1}^{N} x^{b(e)}\right)$$

$$= \frac{1}{N}\sum_{e=1}^{N}\mathbb{E}(x^{b(e)}) = \frac{1}{N}\sum_{e=1}^{N} x^b = \frac{N}{N}x^b = x^b$$

$$\mathbb{E}(\bar{x}^b) - x^b = 0$$

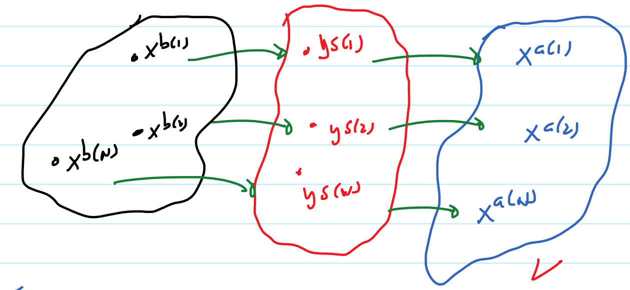
$x^b \approx \bar{x}^b$ $\quad$ $P^b \approx B$ $\qquad$ $x^a \approx \bar{x}^a$ $\quad$ $P^a \approx A$

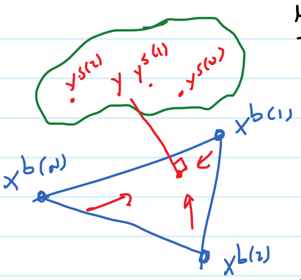
x^b $\qquad$ y $\qquad$ x^a

(F) EnKF $\quad$ GEIR EVENSEN

$\hookrightarrow \quad x^b \approx \bar{x}^b \qquad B \approx P^b$

MUESTRALES





$$x^{a(e)} = [P^{b^{-1}} + H^T R^{-1} H]^{-1} [P^{b^{-1}} x^{b(e)} + H^T R^{-1} y]$$

$$x^{a(e)} = [P^{b^{-1}} + H^T R^{-1} H]^{-1} [P^{b^{-1}} x^{b(e)} + H^T R^{-1} y^{s(e)}]$$

SYNTHETIC OBSERVATION

$X^a \in \mathbb{R}^{n \times N}$ $\quad$ $\in \mathbb{R}^{n \times n}$ $\quad$ $\mathbb{R}^{m \times m}$ $\quad$ 3DVAR $\quad$ $\in \mathbb{R}^{n \times N}$

$y - Hx \sim N(0, R)$

$1 \le e \le N$

$$X^a = [P^{b^{-1}} + H^T R^{-1} H]^{-1} [P^{b^{-1}} X^b + H^T R^{-1} Y^s] \Rightarrow y^{s(e)} \sim N(y, R)$$

$\mathbb{R}^{m \times N} \Rightarrow y^{s(e)} = y + \varepsilon_{obs} \quad \varepsilon_{obs} \sim N(0, R)$

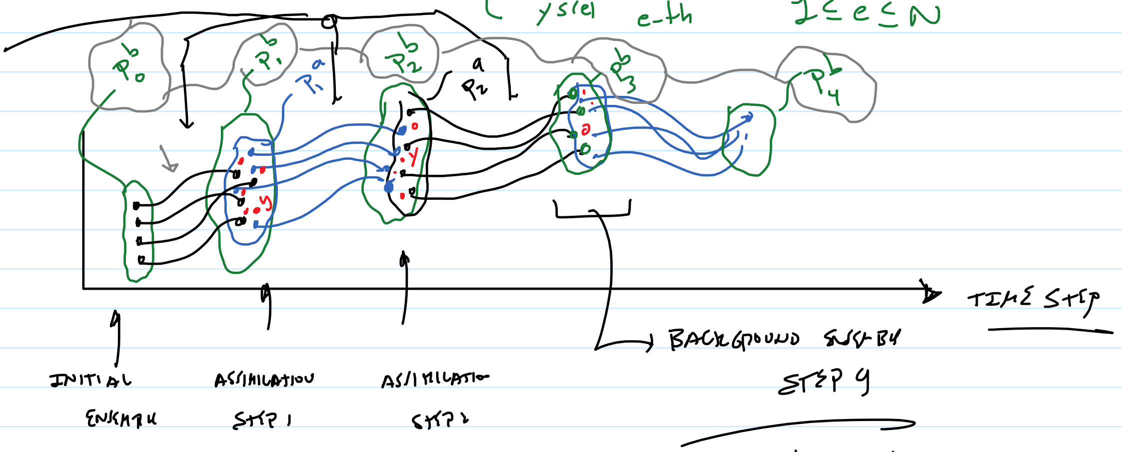
$$X^a = [I + \cdots H^T]^{-1} [I^{-1} X^b + H^T R^{-1} y^s] \Rightarrow y^{s(e)} \sim N(y, R)$$

e-th $x^{a(e)}$ e-th $x^{b(e)}$ $\in \mathbb{R}^{n \times N} \Rightarrow y^{s(e)} = y + \varepsilon_{obs} \quad \varepsilon_{obs} \sim N(0, R)$ $\Leftrightarrow$

$y^{s(e)}$ e-th $1 \le e \le N$

FORECAST

$\underline{\underline{\mu(x)}}$



① SEQUENTIAL → SAMPLES → NUMERICAL MODEL → $\bar{x}^b \approx x^b$, $P^b \approx B$

→ GAUSSIAN ASSUMPTION

⑦ ENSEMBLE KALMAN FILTER

$$P^a = [P^{b^{-1}} + H^T R^{-1} H]^{-1}$$

① $$X^a = [P^{b^{-1}} + H^T R^{-1} H]^{-1} [P^{b^{-1}} X^b + H^T R^{-1} y^s]$$

② $$X^a = X^b + [P^{b^{-1}} + H^T R^{-1} H]^{-1} H^T R^{-1} D \qquad D = y^s - H X^b$$

$$X^a = X^b + P^b H^T [R + H P^b H^T]^{-1} D \quad (*)$$

$$X^a = P^a [P^{b^{-1}} X^b + H^T R^{-1} y^s]$$

$$X^a = X^b + P^a H^T R^{-1} D$$

$\Rightarrow$ ② → ① $$X^a = I X^b + \overbrace{[P^{b^{-1}} + H^T R^{-1} H]^{-1}}^{P^a} H^T R^{-1} D$$

$$\Rightarrow X^a = \underbrace{P^a \cdot P^{a^{-1}}}_{I} X^b + P^a H^T R^{-1} D$$

① ② EQUIV

$$\Rightarrow X^a = P^a [P^{a^{-1}} X^b + H^T R^{-1} [y^s - H X^b]]$$

$$\Rightarrow X^a = P^a [P^{b^{-1}} X^b + H^T R^{-1} H P^b + H^T R^{-1} y^s - H^T R^{-1} H X^b]$$

$$\Rightarrow \quad x^a = P^a \left[P^{b^{-1}} x^b + H^T R^{-1} H P^b + \underbrace{H^T R^{-1} y^s - H^T R^{-1} H x^b}_{0 \in \mathbb{R}^{n \times n}} \right]$$

$$\Rightarrow \quad x^a = P^a \left[P^{b^{-1}} x^b + H^T R^{-1} y^s \right]$$

$$= \left[P^{b^{-1}} + H^T R^{-1} H \right]^{-1} \left[P^{b^{-1}} x^b + H^T R^{-1} y^s \right]$$

WOODBURY MATRIX IDENTITY

$$(+) \quad \left[A + UCV \right]^{-1} = A^{-1} - A^{-1} U \left(C^{-1} + V A^{-1} U \right)^{-1} V A^{-1}$$

$V = U^T$

$$(+) \quad \left[P^{b^{-1}} + H^T R^{-1} H \right]^{-1} = P^b - P^b H^T \left(R + H P^b H^T \right)^{-1} H P^b \qquad (*) \ \checkmark$$

$$\Rightarrow \quad x^a = \left[P^{b^{-1}} + H^T R^{-1} H \right]^{-1} \left[P^{b^{-1}} x^b + H^T R^{-1} y^s \right]$$

$$= \left[P^b - P^b H^T \left(R + H P^b H^T \right)^{-1} H P^b \right] \left[P^{b^{-1}} x^b + H^T R^{-1} y^s \right]$$

$$= x^b + P^b H^T R^{-1} y^s - P^b H^T \left(R + H P^b H^T \right)^{-1} H x^b$$
$$- P^b H^T \left(R + H P^b H^T \right)^{-1} H P^b H^T R^{-1} y^s$$

$$\Updownarrow$$

$$= x^b + P^b H^T \left(R + H P^b H^T \right)^{-1} D \qquad D = y^s - H x^b$$

$$= x^b + P^b H^T \left[R^{-1} y^s - \left(R + H P^b H^T \right)^{-1} H x^b \right.$$
$$\left. - \left(R + H P^b H^T \right)^{-1} H P^b H^T R^{-1} y^s \right]$$

I

$$= x^b + P^b H^T \left[\left(R + H P^b H^T \right)^{-1} \left(R + H P^b H^T \right) R^{-1} y^s - \left(R + H P^b H^T \right)^{-1} H x^b \right.$$
$$\left. - \left(R + H P^b H^T \right)^{-1} H P^b H^T R^{-1} y^s \right]$$

$$= x^b + P^b H^T \left(R + H P^b H^T \right)^{-1} \left[\left(R + H P^b H^T \right) R^{-1} y^s - H x^b - H P^b H^T R^{-1} y^s \right]$$

$$= x^b + P^b H^T (R + HP^bH^T)^{-1} [y^s + HP^bH^T R^{-1} y^s - H x^b - HP^bH^T R^{-1} y^s]$$

$= 0 \in \mathbb{R}^{m \times m}$

$$= x^b + P^b H^T (R + HP^bH^T)^{-1} [y^s - H x^b]$$

$$= x^b + P^b H^T (R + HP^bH^T)^{-1} D$$

$D = y^s - H x^b$

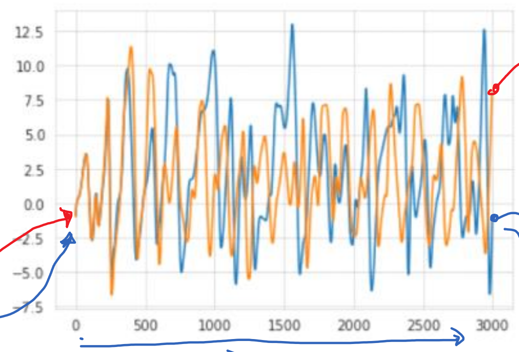
F IMPLEMENTATION

$m(x) = x^2 \Rightarrow$ $1 \Rightarrow m(1) = 1$ $2 \Rightarrow m(2) = 4$

$m(\ ) = -1$

$m(-1) = 1$

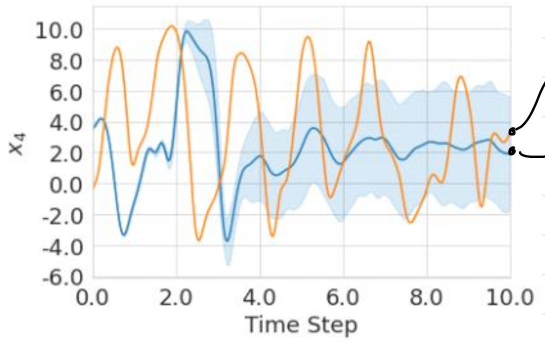
```python
xb = xt+0.05*np.random.randn(n);
x = odeint(lorenz96, xb, t);
xb = x[-1,:]; #Background state (initial)
xs = odeint(lorenz96, xt, t);
xt = xs[-1,:]; #xb(k) - xt(k) en ambos k es el mismo

sns.set_style('whitegrid')
plt.plot(x[:,1]);
plt.plot(xs[:,1]);
```



0 30

x^b IS CONSISTENT WITH MODEL DYNAMICS

$ts = 0.01$



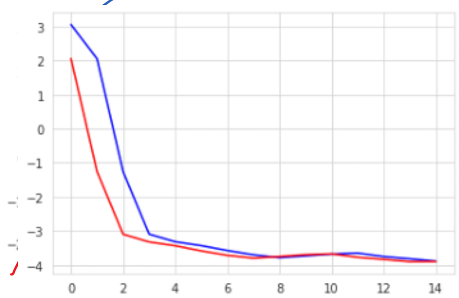
x^*

$\bar{x}^b$] P^b

$y^s - Hx^b$

$X^a = x^b + \underbrace{P^b H^T Z}$ ✓ $\qquad \underbrace{[R + HP^bH^T] \, Z = D}$ ✓

BACKGROUND ERROR



ANALYSIS

$\underline{\| \bar{x}^b_k - x^*_k \|}$ $\qquad$ $\underline{\| \bar{x}^a_k - x^*_k \|}$

```
np.random.seed(seed=10);

time_step = 0.01;
```

→ TIME BTW OBSERVATIONS

```
T = 15; #Number of time steps to simulate

I = np.eye(n,n);

#Initial variables for the tests.
xt_k = xt.copy();
```

$x_0^* \in \mathbb{R}^{n \times 1}$

```
XB_k = XB.copy().transpose();
```

$X_0^b \in \mathbb{R}^{n \times N}$

```
#Number of observed components
p = 0.8;
m = int(round(p*n));
sig_obs = 0.01;
R_k = sig_obs*np.eye(m,m);
```

$**2$

$R_k = \sigma^2 \cdot I_{m \times m}$

```
errorb_k = np.zeros(T);
errora_k = np.zeros(T);


for k in range(0,T):
```

$0 \to T$

```
  print(f'Assimilation Cycle {k}');

  #Prior parameters
  xb_k = np.mean(XB_k,1);
  PB_k = np.cov(XB_k);
```

$\bar{x}^b_k = \frac{1}{N} \sum_{e=1}^{N} x_k^{b(e)}$

$cov(x^b_k) = P^b_k$

```
#Prior error
errorb_k[k] = np.linalg.norm(xt_k-xb_k);

#Observation at time step k - we are simulating an observation
obs_comp = np.random.permutation(n);
obs_comp = obs_comp[0:m];
H_k = I[obs_comp,:];
err_obs_k = sig_obs*np.random.randn(m);
y_k = H_k@xt_k + err_obs_k;

#Synthetic observations
Eobs_k = sig_obs*np.random.randn(m,N);
Yobs_k = np.outer(y_k,np.ones(N)) + Eobs_k;

#Matrix of innovation ob the synthetic observations
D_k = Yobs_k-H_k@XB_k;

#Innovation matrix
IN_k = R_k + H_k@PB_k@H_k.transpose();
Z_k = np.linalg.solve(IN_k,D_k);

#Analysis ensemble (posterior ensemble)
XA_k = XB_k + PB_k@H_k.transpose()@Z_k;
xa_k = np.mean(XA_k,1);

#Posterior error
errora_k[k] = np.linalg.norm(xt_k-xa_k);
```

$\| x_k^t - x_k^b \|$

$[0, 1, -, n]$

$\underbrace{[1, 2, n, 0, 5, -]}_{m}$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$e_k = \sigma \cdot \mu \quad \mu = \sim \mathcal{N}(0, I)$

$\sigma \mu = \mathcal{N}(0, \sigma^2 I)$

$y_k = H x^t + e_k$

$y^s = \underbrace{[y, y, \dots y]}_{N} + \sigma \cdot \underbrace{[\quad]}_{\mathcal{N}(0,1)}$

$y \cdot 1_N^T$

outer product

$\sigma \; \mathcal{N}(0, \sigma^2)$

$\sigma^2 \underbrace{[\quad]}_{\mathcal{N}(0,1)}$

$\sigma^2 \Rightarrow \mathcal{N}(0, \sigma^2)$

$R + H P^b H^T$

$D_k = y_k^s - H_k \cdot X_k^b$

$[R + H P^b H^T] Z_k = D_k$

$X_k^a = X_k^b + P_k^b H_k^T Z_k$

$\overline{x}_k^a = \frac{1}{N} \sum_{e=1}^{N} x^{a(e)} \in \mathbb{R}^{n \times 1}$